

# $L^2$ -EXTENSION OF $\bar{\partial}$ -CLOSED FORMS

BO BERNDTSSON

ABSTRACT. Generalizing and strengthening a recent result of Koziarz, we prove a version of the Ohsawa-Takegoshi-Manivel theorem for  $\bar{\partial}$ -closed forms.

## 1. INTRODUCTION

The celebrated Ohsawa-Takegoshi-Manivel extension theorem, [7],[6] gives optimal conditions for the extension of holomorphic sections of line bundles from a divisor to the ambient space. In Manivel's article, [6], it is stated that a completely parallel result holds for smooth  $\bar{\partial}$ -closed forms of higher degree. There is however a problem in the proof of this in [6] which is connected with the regularity of solutions of certain  $\bar{\partial}$ -equations with singular weights. This problem is also discussed in [4], where a strategy towards its solution is put forward.

Recently, an at least moral solution of this problem was given by Koziarz, [5]. Instead of looking at the extension of individual forms, Koziarz considered the extension of cohomology classes, i.e. extended closed forms up to a  $\bar{\partial}$ -exact error. This formulation is actually more natural than the original problem since cohomology classes have well defined restrictions on divisors, whereas  $\bar{\partial}$ -closed forms restrict only if a somewhat artificial condition of smoothness is imposed. Koziarz's method is inspired by work of Siu, [8], and consists in representing cohomology classes by Čech cocycles. These cocycles consist of holomorphic objects for which the available machinery works better.

The purpose of this note is twofold. First we will prove a simple proposition saying that a smooth  $\bar{\partial}$ -exact form on a divisor can always be extended to a closed form with arbitrary small  $L^2$ -norm in the ambient space. (This property characterizes exact forms.) This means that Koziarz's theorem on the extension up to an exact error actually gives a solution to the original problem on extension of closed forms. Second, we will give an alternative proof of Koziarz's theorem, following the method in [2]. The advantage with this alternative proof is that it gives an absolute constant for the extension, whereas in Koziarz's theorem the constant depended on the manifold and the divisor. Moreover, the curvature conditions that guarantee extendability are shown to be somewhat more liberal for forms of higher degree than for holomorphic sections. Finally, the proof exhibits the significance of extension of cohomology classes in a seemingly interesting way.

Let us comment a little bit more on this. If  $u$  is a holomorphic section of  $K_\Delta + L$  over a divisor  $\Delta$ , the method in [2], see also [1], consists in solving the equation

$$\bar{\partial}v = u \wedge [\Delta] := g.$$

The right hand side here is not a  $L^2$ -form but a current, but nevertheless it turns out that  $L^2$ -methods can be used here. One cannot however get a solution  $v$  in  $L^2$ . If the divisor  $\Delta$  is defined

by a section  $s$  of some line bundle  $S$  over the ambient manifold  $X$ , the solution of the extension problem is  $sv$ , so what we want is an  $L^2$ -estimate for  $sv$ . Dually, (and formally!) this corresponds to an estimate for smooth testforms  $\alpha$  like

$$(1.1) \quad |\langle g, \alpha \rangle|^2 \leq C \int |\bar{\partial}^* \alpha|^2 / |s|_\psi^2$$

(where  $\psi$  is some metric on  $S$ ). But this dual formulation is only formal. The fact that the weight  $|s|^{-2}$  is nonintegrable causes a problem in the functional analysis involved since all smooth test forms do not have finite norm with respect to this weight. This problem can be circumvented if we instead prove a stronger estimate

$$(1.2) \quad |\langle g, \alpha \rangle|^2 \leq C \int |\bar{\partial}^* \alpha|^2 / |s|_\psi^r$$

where  $r < 2$ . Then the functional analytic difficulty disappears and one even gets a stronger result than is asked for.

We now want to follow the same route for forms of higher degree. Booth estimates (1.1) and (1.2) can then be proved in much the same manner as for holomorphic sections. As in the case of holomorphic sections, the best thing would be to use (1.2), since that is a bona fide dual formulation of the  $\bar{\partial}$ -problem. But this causes problems with regularity. One would then need to discuss regularity properties in  $L^2$ -spaces with singular weights, which leads back to the original problem with Manivel's argument. We therefore choose to work with (1.1) instead. Then the regularity problems disappear since we can go back and forth between estimates with the singular weight  $|s|^{-2}$  and estimates without that weight by multiplying and dividing with  $s$ . The price we have to pay for this is that (1.1) is no longer a dual formulation of the  $\bar{\partial}$ -estimate, and so not a dual formulation of the extension problem. But, miraculously, it turns out to be a dual formulation of the extension of cohomology classes, and this is what makes the scheme work.

In this paper we will suppose all the time that  $X$  is a compact Kähler manifold. Maybe the same arguments could be pushed to non compact situations, but the compactness assumption simplifies and makes the argument a little bit simpler than in [2]

## 2. $\bar{\partial}$ -EXACT FORMS

In this section we discuss the extension of  $\bar{\partial}$ -exact forms. The main point is the following proposition.

**Proposition 2.1.** *Let  $X$  be an  $n$ -dimensional compact complex manifold, and let  $\Delta$  be a smooth divisor in  $X$ . Let  $L$  be a holomorphic line bundle over  $X$ . Let  $u$  be a smooth  $\bar{\partial}$ -closed  $L$ -valued  $(0, q)$ -form on  $\Delta$ ,  $q \geq 1$ . Then  $u$  is  $\bar{\partial}$ -exact on  $\Delta$  if and only if, for any  $\epsilon > 0$ , there is an extension,  $U$ , of  $u$  to  $X$  with  $L^2$ -norm smaller than  $\epsilon$ .*

Here  $L^2$ -norms are taken with respect to some smooth metric and some arbitrary smooth volume form. In the proof we use the next lemma.

**Lemma 2.2.** *There is a sequence of cutoff-functions  $\rho_\epsilon$  such that*

1. The sets where  $\rho_\epsilon(z) = 1$  are neighbourhoods of  $\Delta$  shrinking to  $\Delta$ , and the sets where  $\rho_\epsilon(z) = 0$  increase to  $X \setminus \Delta$ .
2.  $\|\bar{\partial}\rho_\epsilon\|$  goes to zero with  $\epsilon$ .

*Proof.* Let first the dimension be 1 and take  $X$  to be the unit disk and  $\Delta$  to be the origin. The main point is that there is a complete Kahler metric on the punctured disk,  $\omega$ , which gives  $\{|z| < 1/2\}$  finite area. Indeed, the Poincare metric

$$\omega = idz \wedge d\bar{z} / (|z|^2 (\log |z|)^2)$$

has this property. Completeness implies that there is some realvalued function near the origin,  $\rho$ , such that  $\rho(z)$  tends to infinity when  $z$  tends to zero and

$$i\partial\rho \wedge \bar{\partial}\rho \leq \omega.$$

Explicitly,  $\rho(z) = \log \log(1/|z|)$  will do. Define functions  $\chi_k(x)$  on the positive halfaxis, equal to 0 when  $x < k$ , to 1 when  $x > k + 1$ , and having  $\chi'_k$  bounded. Then put

$$\rho_\epsilon = \chi_{1/\epsilon} \circ \rho.$$

Then 1 is clear and 2 follows by dominated convergence since

$$\int_{|z| < 1/2} i\partial\rho_\epsilon \wedge \bar{\partial}\rho_\epsilon \leq \int_{|z| < 1/2} \chi'_\epsilon \omega.$$

The general case is basically the same. We can cover  $\Delta$  by a finite number of coordinate neighbourhoods, inside which  $\Delta$  is defined by the equation  $z_1 = 0$ . Then take  $\rho_\epsilon(z_1)$  with  $\rho_\epsilon$  defined as above and piece together with a partition of unity.  $\square$

With this we can turn to the proof of the proposition. Assume first that  $u = \bar{\partial}v$  on  $\Delta$  with  $v$  smooth. We extend  $v$  to  $X$  in an arbitrary way and let

$$U_\epsilon = \bar{\partial}(\rho_\epsilon v).$$

By the Lemma this a  $\bar{\partial}$ -closed, or even exact, extension of  $u$  with  $L^2$ -norm going to zero with  $\epsilon$ .

For the converse, assume there are some  $\bar{\partial}$ -closed extensions,  $U_\epsilon$ , with  $L^2$ -norms going to zero. Let  $\mathcal{U}_\epsilon$  be the harmonic representative of the cohomology class  $[U_\epsilon]$ . The norms of the harmonic representatives are smaller, so they go to zero too. Now, the space of harmonic forms is finite dimensional, so all norms are equivalent. Hence the supnorms of  $\mathcal{U}_\epsilon$  also go to zero, so the restrictions of  $\mathcal{U}_\epsilon$  to  $\Delta$  also go to zero. Since on  $\Delta$ ,  $u - \mathcal{U}_\epsilon$  is exact, it follows that  $u$  lies in the closure of the space of exact forms. But  $\bar{\partial}$  has closed range on a compact manifold, so  $u$  must be exact.

### 3. $\bar{\partial}$ -CLOSED FORMS

In this section we adapt the argument in [2] to forms of higher degree. We will use the residue formulation of the extension problem and the set up is as follows.

$X$  is a compact Kähler manifold, with Kähler form  $\omega$  and  $L$  is a holomorphic line bundle over  $X$ .  $\Delta$  is a smooth divisor in  $X$ , given as  $\Delta = s^{-1}(0)$ , with  $s$  a holomorphic section of a line

bundle  $S$ . Let  $u$  be a smooth  $L$ -valued  $\bar{\partial}$ -closed  $(n-1, q)$ -form on  $\Delta$ . We want to find a smooth  $L$ -valued  $\bar{\partial}$ -closed  $(n, q)$ -form,  $U$ , on  $X$ , such that

$$(3.1) \quad U = ds \wedge u$$

on  $\Delta$ . Note that  $u$  could alternately be interpreted as a  $(0, q)$ -form on  $\Delta$  with values in  $K_\Delta + L$ . By the adjunction isomorphism

$$u \mapsto ds \wedge u$$

between  $K_\Delta$  and  $(K_X + S)|_Y$  this means that we extend a  $(0, q)$ -form with values in

$$(3.2) \quad F := K_X + S + L$$

to a form with values in  $F$ .

**Theorem 3.1.** *Assume that  $\phi$  is a smooth metric on  $L$  and that  $\psi$  is a smooth metric on  $S$  such that*

$$i\partial\bar{\partial}\phi \wedge \omega^q \geq \epsilon i\partial\bar{\partial}\psi \wedge \omega^q$$

and

$$i\partial\bar{\partial}\phi \wedge \omega^q \geq 0.$$

Assume moreover the normalizing inequality

$$\log |s|^2 e^{-\psi} \leq -1/\epsilon.$$

Let  $u$  be a smooth  $\bar{\partial}$ -closed  $(n-1, q)$ -form with values in  $L$  over  $\Delta$ . Then there is a  $\bar{\partial}$ -closed  $(n, q)$ -form,  $U$ , with values in  $S + L$  over  $X$  such that

$$U = ds \wedge u$$

on  $\Delta$  and

$$\int_X |U|^2 e^{-\phi-\psi} dV_X \leq C_0 \int_\Delta |u|^2 e^{-\phi} dV_Y$$

where  $C_0$  is an absolute constant. The norms and the volume forms are defined by the Kähler form  $\omega$ .

The arguments starts with the observation that if  $U$  satisfies the conclusion of the theorem, and if  $v := U/s$ , then  $v$  has values in  $K_X + L$  and solves

$$(3.3) \quad \bar{\partial}v = \bar{\partial}(1/s) \wedge ds \wedge u = cu \wedge [\Delta]$$

where  $[\Delta]$  is the current of integration on  $\Delta$ . Conversely, let  $v$  solve (3.3) and assume that  $U := sv$  is smooth. On  $\Delta$  we can write  $U = ds \wedge \tilde{u}$  by the adjunction isomorphism. Then

$$\bar{\partial}v = \bar{\partial}(1/s) \wedge ds \wedge \tilde{u} = c\tilde{u} \wedge [\Delta].$$

Hence  $\tilde{u} = u$  on  $\Delta$ , so  $U$  solves the extension problem.

We now try (and fail!) to solve this  $\bar{\partial}$ -problem and start to give it a dual formulation. Let

$$f := u \wedge [\Delta],$$

a current with measure coefficients, concentrated on  $\Delta$  and of bidegree  $(n, q+1)$ . The proof of the next lemma will be postponed to the end of the section.

**Lemma 3.2. ( The basic estimate )** Assume, in addition to the assumptions in Theorem 3.1, that

$$\|u\|_{\Delta}^2 \leq 1$$

Then, for any smooth  $L$ -valued  $(n, q)$ -form  $\alpha$  on  $X$

$$|\langle f, \alpha \rangle|^2 \leq C_0 \int_X \frac{|\bar{\partial}_\phi^* \alpha|^2}{|s|^2 e^{-\psi}} e^{-\phi}.$$

The norm  $\|\cdot\|_{\Delta}$  here is the  $L^2$ -norm defined by the Kähler form  $\omega$  and the metric  $\phi$  on  $L$ .

Now consider the conjugate linear functional

$$R(\bar{\partial}_\phi^* \alpha) = \langle f, \alpha \rangle$$

defined on the space

$$E := \{\bar{\partial}_\phi^* \alpha; \alpha \text{ smooth}\}.$$

By the lemma,  $R$  is bounded by the norm

$$\int_X \frac{|\bar{\partial}_\phi^* \alpha|^2}{|s|^2 e^{-\psi}} e^{-\phi}$$

on the subspace  $E_0$  of elements of  $E$  such that this norm is finite. Clearly, this subspace consists of forms  $\bar{\partial}_\phi^* \alpha$  that vanish on  $\Delta$ . By the Riesz representation theorem, there is a form  $w$  such that

$$R(\bar{\partial}_\phi^* \alpha) = \int_X \frac{w \cdot \overline{\bar{\partial}_\phi^* \alpha}}{|s|^2 e^{-\psi}} e^{-\phi}$$

for all  $\alpha$  with  $\bar{\partial}_\phi^* \alpha = 0$  on  $\Delta$ . Moreover,  $w$  can be taken to satisfy

$$\int_X \frac{|w|^2}{|s|^2 e^{-\psi}} e^{-\phi} \leq C_0.$$

Substitute

$$v = w/(|s|^2 e^{-\psi}).$$

Then

$$(3.4) \quad \langle f, \alpha \rangle = R(\bar{\partial}_\phi^* \alpha) = \int_X v \cdot \overline{\bar{\partial}_\phi^* \alpha} e^{-\phi}$$

and

$$(3.5) \quad \int_X |v|^2 |s|^2 e^{-\phi-\psi} \leq C_0.$$

Notice that this *does not mean* that  $\bar{\partial} v = f$  since we only know that (3.4) holds for  $\alpha$  with  $\bar{\partial}_\phi^* \alpha = 0$  on  $\Delta$ .

In order to get smoothness we now choose  $v$  with minimal norm defined in (3.5), and the first objective is to check that there is a minimizer.

**Lemma 3.3.** Assume that  $v_k$  is a sequence of forms such that

$$\langle f, \alpha \rangle = \int_X v_k \cdot \overline{\partial_\phi^* \alpha} e^{-\phi}$$

for all  $\alpha$  with  $\bar{\partial}_\phi^* \alpha = 0$  on the divisor. Assume also that

$$\int_X |v - v_k|^2 |s|^2 e^{-\phi-\psi} \rightarrow 0$$

for some  $v$  satisfying

$$\int_X |v|^2 |s|^2 e^{-\phi-\psi} < \infty.$$

Then

$$\langle f, \alpha \rangle = \int_X v \cdot \overline{\partial_\phi^* \alpha} e^{-\phi}$$

for all  $\alpha$  with  $\bar{\partial}_\phi^* \alpha = 0$  on the divisor.

This means that the affine space of forms  $v$  that satisfy (3.4) is closed for the norm in (3.5), so it has an element of minimal norm. The proof of the lemma is clear since

$$\left| \int_X (v - v_k) \cdot \overline{\partial_\phi^* \alpha} e^{-\phi-\psi} \right|^2 \leq \int_X |v - v_k|^2 |s|^2 e^{-\phi-\psi} \int_X \frac{|\bar{\partial}_\phi^* \alpha|^2}{|s|^2 e^{-\psi}} e^{-\phi}.$$

The next point is to see that if  $v$  is a minimizer, then  $sv$  is a harmonic form, hence smooth.

**Lemma 3.4.** Assume that  $v$  minimizes the norm in (3.5) among all solutions to (3.4). Then  $\bar{\partial}_{\phi+\psi}^*(sv) = 0$ .

*Proof.* If  $v$  is a minimizer then

$$\int_X |v|^2 |s|^2 e^{-\phi-\psi} \leq \int_X |v - \bar{\partial}u|^2 |s|^2 e^{-\phi-\psi}$$

for all smooth  $u$ . This means that

$$\int_X sv \cdot \overline{\partial sue} e^{-\phi-\psi} = 0.$$

Hence  $\bar{\partial}_{\phi+\psi}^* sv = 0$  at least outside of  $\Delta$ . But  $sv$  has finite  $L^2$ -norm so a divisor is removable for this equation. (A  $\bar{\partial}^*$ -equation for a form is a  $\bar{\partial}$ -equation for  $*$  of the form.)  $\square$

Finally we have

**Lemma 3.5.**

$$\bar{\partial}(sv) = 0.$$

*Proof.* Since  $sv$  takes values in  $L + S$ , we have to check that

$$\langle sv, \bar{\partial}_{\phi+\psi}^* \xi \rangle_{\phi+\psi} = 0$$

for any smooth  $(n, q)$ -form  $\xi$  with values in  $L + S$ . For this, note first that

$$\sigma := \bar{s}e^{-\psi} = |s|^2 e^{-\psi} / s$$

is a smooth section with values in  $-S$ , which vanishes on  $\Delta$ . Therefore  $\alpha_\xi := \sigma\xi$  is  $L$ -valued and vanishes on  $\Delta$ . Moreover, one easily checks that

$$(3.6) \quad \bar{s}\bar{\partial}_{\phi+\psi}^*\xi = e^\psi\bar{\partial}_\phi^*\alpha_\xi.$$

Hence

$$\langle sv, \bar{\partial}_{\phi+\psi}^*\xi \rangle_{\phi+\psi} = \langle v, \bar{\partial}_\phi^*\alpha_\xi \rangle_\phi = \langle f, \alpha_\xi \rangle_\phi,$$

where the last equality follows from (3.4). We are allowed to apply (3.4) because  $\bar{\partial}_\phi^*\alpha_\xi$  is zero on  $\Delta$  by (3.6). Since  $f$  is supported on  $\Delta$  where  $\alpha_\xi$  vanishes,  $\langle f, \alpha_\xi \rangle_\phi$  equals zero, and we are done.  $\square$

All in all we have now seen that  $U := sv$  is harmonic and therefore smooth, if  $v$  is the minimal solution of the dual problem. What remains is to investigate the behaviour of  $U$  on the divisor. Write  $U = ds \wedge \tilde{u}$  on the divisor. Let  $\alpha$  be a smooth  $L$ -valued  $(n, q+1)$ -form such that  $\bar{\partial}_\phi^*\alpha = 0$  on the divisor and write

$$\alpha = \gamma_\alpha \wedge \omega^{q+1}/(q+1)!$$

for some (uniquely determined)  $(n - q - 1, 0)$ -form  $\gamma_\alpha$ . Then for any  $(n, q+1)$ -form  $g$

$$\langle g, \alpha \rangle_\omega \omega^n / n! = g \wedge \bar{\gamma}_\alpha,$$

(see [3] for more on this).

Hence

$$\langle f, \alpha \rangle = \int_X f \wedge \bar{\gamma}_\alpha e^{-\phi} = \int_\Delta u \wedge \bar{\gamma}_\alpha e^{-\phi}.$$

On the other hand, by (3.4) this also equals

$$\int_X v \cdot \bar{\partial}_\phi^*\alpha e^{-\phi} = \int_X U/s \cdot \bar{\partial}_\phi^*\alpha e^{-\phi} = \int_X \bar{\partial}(1/s) \wedge U \wedge \bar{\gamma}_\alpha e^{-\phi} = \int_\Delta \tilde{u} \wedge \bar{\gamma}_\alpha e^{-\phi}.$$

From this we see that

$$(3.7) \quad \int_\Delta u \wedge \bar{\gamma}_\alpha e^{-\phi} = \int_\Delta \tilde{u} \wedge \bar{\gamma}_\alpha e^{-\phi}$$

for all  $\alpha$  such that  $\bar{\partial}_\phi^*\alpha = 0$  on  $\Delta$ . This latter condition is equivalent to saying that

$$\bar{\partial}(\bar{\gamma}_\alpha e^{-\phi}) = 0.$$

Let  $\bar{\gamma}_\alpha e^{-\phi} =: \chi$ . This is a  $(0, n - q - 1)$ -form with values in  $-L$ . Hence

$$(3.8) \quad \int_\Delta (u - \tilde{u}) \wedge \chi = 0$$

for all  $(0, n - q - 1)$ -forms  $\chi$  with values in  $-L$  such that  $\bar{\partial}\chi = 0$  on  $\Delta$ . The  $\bar{\partial}$  operator here is the  $\bar{\partial}$  on  $X$ , but, by the next lemma, the same thing holds if only  $\bar{\partial}_\Delta\chi = 0$ .

**Lemma 3.6.** *Let  $\chi$  be a smooth  $-L$ -valued  $(0, p)$ -form on  $X$  such that  $\bar{\partial}_\Delta\chi = 0$  on  $\Delta$ . Then there is a smooth form on  $X$ ,  $\tilde{\chi}$  such that  $\bar{\partial}_X\tilde{\chi} = 0$  on  $\Delta$  and  $\chi = \tilde{\chi}$  on  $\Delta$ .*

*Proof.* Locally the divisor is given by an equation  $z_1 = 0$  in some local chart. The hypothesis then means that  $\bar{\partial}\chi$  is divisible by  $d\bar{z}_1$ . To get a local extension it therefore suffices to subtract a suitable multiple of  $\bar{z}_1$ , and one then obtains  $\tilde{\chi}$  from a partition of unity.  $\square$

It follows from the lemma that (3.8) holds for any  $\chi$  on  $\Delta$  such that  $\bar{\partial}_\Delta \chi = 0$ . But this means that the difference  $u - \tilde{u}$  is  $\bar{\partial}$ -exact. Hence we have proved Koziarz's theorem that  $u$  can be extended up to an exact error, and the proof of Theorem 3.1 then follows from Proposition 2.1.

All that remains is now to prove Lemma 3.2.

**3.1. Proof of the basic estimate.** This follows closely the proof in [2], and the proof in the compact case is described in [3], and we refer to these notes for more details on the computations that follow.

We first write as above

$$\alpha = \gamma \wedge \omega^{q+1}/(q+1)!$$

so that  $\gamma$  is an  $L$ -valued  $(n - q - 1, 0)$ -form. Then define a scalar valued  $(n - 1, n - 1)$ -form

$$T_\alpha = c_q \gamma \wedge \bar{\gamma} \wedge \omega^q e^{-\phi}/q!$$

where  $c_q$  is a unimodular constant chosen so that  $T_\alpha$  is a positive form. We will prove the basic estimate first assuming that  $\bar{\partial}\alpha = 0$ . In that case it follows from Proposition 3.4.1 in [3] that

$$(3.9) \quad i\bar{\partial}\bar{\partial}T_\alpha \geq -2\operatorname{Re}(\bar{\partial}\bar{\partial}_\phi^*\alpha, \alpha)\omega^n/n! + i\bar{\partial}\bar{\partial}\phi \wedge T_\alpha.$$

Let

$$W := -\log(|s|^2 e^{-\psi}).$$

By the hypothesis in Theorem 3.1,  $W \geq 1/\epsilon$ . Moreover

$$i\bar{\partial}\bar{\partial}W = i\bar{\partial}\bar{\partial}\psi - c[\Delta].$$

Multiply (3.9) by  $W$  and apply Stokes' formula. This gives

$$(3.10) \quad \int_X (W i\bar{\partial}\bar{\partial}\phi \wedge \omega^q - i\bar{\partial}\bar{\partial}\psi \wedge \omega^q)/q! \wedge c_q \gamma \wedge \bar{\gamma} e^{-\phi} + c \int_\Delta c_q \gamma \wedge \bar{\gamma} \wedge \omega^q/q! e^{-\phi} \leq 2\operatorname{Re} \langle \bar{\partial}\bar{\partial}_\phi^* \alpha, W\alpha \rangle.$$

By the hypotheses in Theorem (3.1) the first integral in the left hand side is nonnegative, so we get

$$c \int_\Delta c_q \gamma \wedge \bar{\gamma} \wedge \omega^q/q! e^{-\phi} \leq 2\operatorname{Re} \langle \bar{\partial}\bar{\partial}_\phi^* \alpha, W\alpha \rangle.$$

On the other hand

$$|\langle f, \alpha \rangle|^2 = \left| \int_X f \wedge \bar{\gamma} e^{-\phi} \right|^2 = \left| \int_\Delta u \wedge \bar{\gamma} e^{-\phi} \right|^2.$$

By the Cauchy inequality we get, since by assumption  $\|u\|_\Delta \leq 1$  that

$$|\langle f, \alpha \rangle|^2 \leq \|\gamma\|_\Delta^2 = \int_\Delta c_q \gamma \wedge \bar{\gamma} \wedge \omega^q/q! e^{-\phi} \leq 2c^{-1} \operatorname{Re} \langle \bar{\partial}\bar{\partial}_\phi^* \alpha, W\alpha \rangle.$$



The right hand side equals

$$2 \int_X W |\bar{\partial}_\phi^* \alpha|^2 e^{-\phi} - 2 \operatorname{Re} \langle \bar{\partial} W \wedge \bar{\partial}_\phi^* \alpha, \alpha \rangle.$$

The first term is obviously OK since  $W \leq e^W$ . For the second term we write

$$II := \langle \bar{\partial} W \wedge \bar{\partial}_\phi^* \alpha, \alpha \rangle = \int_X \bar{\partial} W \wedge \bar{\partial}_\phi^* \alpha \wedge \bar{\gamma}_\alpha e^{-\phi}.$$

By Cauchy's inequality

$$2|II| \leq \int_X \frac{|\bar{\partial}_\phi^* \alpha|^2}{|s|^2 e^{-\psi}} e^{-\phi} + c_q \int_X e^{-W} \bar{\partial} W \wedge \gamma_\alpha \wedge \bar{\gamma}_\alpha \wedge \omega^q / q! e^{-\phi}.$$

It is only the last term that we need to worry about. Let

$$W_1 = (1 - e^{-W}).$$

Then  $0 < W_1 < 1$  and

$$i \bar{\partial} \bar{\partial} W_1 = -e^{-W} i \bar{\partial} W \wedge \bar{\partial} W.$$

We now repeat the same argument as above, but with  $W$  replaced by  $W_1$ . The result is

$$c_q \int_X e^{-W} \bar{\partial} W \wedge \bar{\partial} W \wedge \gamma_\alpha \wedge \bar{\gamma}_\alpha \wedge \omega^q / q! e^{-\phi} \leq 2 \int_X W_1 |\bar{\partial}_\phi^* \alpha|^2 e^{-\phi} - 2 \operatorname{Re} \langle \bar{\partial} W_1 \wedge \bar{\partial}_\phi^* \alpha, \alpha \rangle.$$

The first term is controlled since  $W_1 < 1$  and the second term can easily be absorbed in the left hand side. This completes the proof of the basic estimate in case  $\bar{\partial} \alpha = 0$ .

The general case is easily reduced to this special case. We decompose

$$\alpha = \alpha^1 + \alpha^2$$

where  $\alpha^1$  is  $\bar{\partial}$ -closed and  $\alpha^2$  is orthogonal to the space of  $\bar{\partial}$ -closed forms. Then in particular  $\alpha^2$  is orthogonal to  $\bar{\partial}$ -exact forms, so  $\bar{\partial}_\phi^* \alpha^2 = 0$ . Hence  $\alpha^1$  satisfies  $\bar{\partial} \alpha^1 = 0$  and  $\bar{\partial}_\phi^* \alpha^1 = \bar{\partial}_\phi^* \alpha$ . This means, by elliptic regularity that  $\alpha^1$ , and therefore  $\alpha^2$  are both smooth. Now we claim that both sides in the basic estimate are unchanged if we replace  $\alpha$  by  $\alpha^1$ . Since we know the basic estimate holds for  $\alpha^1$  this is all we need. That the right hand side is unchanged we have already seen. That the left hand side is unchanged follows since  $f$  is closed and  $\alpha^2$  is orthogonal to closed forms. There is a minor problem here, coming from the fact that  $f$  is not an  $L^2$ -form. However,  $f$  is cohomologous to a smooth form

$$f = f_{\text{smooth}} + \bar{\partial} g$$

and this proves the claim since  $\alpha^2$  is smooth and satisfies  $\bar{\partial}_\phi^* \alpha^2 = 0$ .

## REFERENCES

- [1] AMAR, E: *Extension de fonctions holomorphes et courants*, Bull. Sciences Mathématiques, 2ème série, 107 (1983), 25-48.
- [2] BERNDTSSON, B: *The extension theorem of Ohsawa-Takegoshi and the theorem of Donnelly-Fefferman.*, Ann. Inst. Fourier (Grenoble) 46 (1996), no. 4, 1083 -1094 .
- [3] BERNDTSSON, B: *An Introduction to things  $\bar{\partial}$* , Lecture Notes from the PCMI .

- [4] DEMAILLY, J-P: *On the Ohsawa-Takegoshi-Manivel  $L^2$  extension theorem*, Complex analysis and geometry (Paris, 1997). Prog. Math., vol. 188, pp. 47 -82.
- [5] KOZIARZ, V: *Extensions with estimates of cohomology classes.* , Manuscripta Math. 134 (2011), no. 1-2, 43-58 .
- [6] MANIVEL, L: *Un théorème de prolongement  $L^2$  de sections holomorphes d'un fibré hermitien* , Math. Z. 212 (1993), no. 1, 107 -122 .
- [7] OHSAWA, T AND TAKEGSOHI, K: *On the extension of  $L^2$  holomorphic functions.* , Math. Z. 195 (1987), no. 2, 197 - 204. .
- [8] SIU, Y-T: *A vanishing theorem for semipositive line bundles over non-Kähler manifolds.*, J. Diff. Geom. 19, 431 - 452.

B BERNDTSSON :DEPARTMENT OF MATHEMATICS, CHALMERS UNIVERSITY OF TECHNOLOGY, AND DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GÖTEBORG, S-412 96 GÖTEBORG, SWEDEN,,  
*E-mail address:*    bob@math.chalmers.se